YEDİTEPE UNIVERSITY

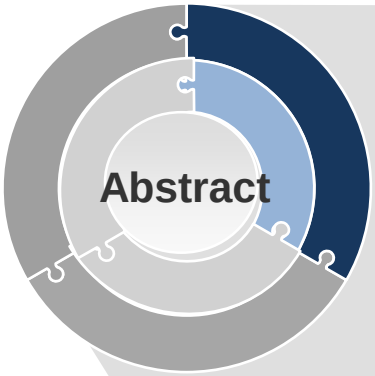



The Econometrics of Efficient Portfolios

Dr. Caner Özdurak

Septembre 2019

An efficient portfolio maximizes the expected utility of the future wealth



- It is well known that the standard mean variance approach can be inappropriate when return distributions feature skewness, fat tails or multimodels.
- This paper presents an analysis of the efficiency frontier, formed by a set of efficient portfolios corresponding to a parameterized class of utility functions.
- Skewed, leptokurtic or multimodal distributions are frequently encountered in practice, especially when derivative assets are included in the portfolio.
- **In this situation, the mean variance approach is not appropriate**

Introduction

- An alternative to the mean variance framework is based on the maximization of an expected utility of future wealth but, given the simplicity of the mean variance approach, efforts have been made in order to see in what circumstances it was equivalent to an expected utility approach.
- In fact, the arguments for adopting the mean variance approach and the normality assumption for portfolio management and statistical inference are weak and mainly based on their simplicity of implementation.

- The aim of this paper is to propose econometric methods, which are valid for any kind of utility function and return distribution assuming that the returns are iid, but not necessarily normal.
- For instance, the Basle Committee suggested a historical simulation of the Value at Risk to account for asymmetry and tails, instead of a mean variance approach, which would strongly underestimate the required capital.

Main results in modern portfolio theory

- First, the Markowitz approach is presented. Then the capital asset pricing model is derived and its empirical testability is discussed. Afterwards Neumann-Morgenstern utility theory is applied to the portfolio problem. Mean-variance analysis is rather unsatisfactory on theoretical grounds. Fortunately, NM utility theory can be applied to the portfolio problem.

- In this context in our reference paper, the portfolio problem which is identified very simply in the modern portfolio theory, is tried to be extended as multifund seperation hypothesis.

- Based on two fund seperation hypothesis K-fund seperation hypothesis is also revisited to solve the optimizatiom problem in a more efficient way than modern portfolio theory which operates mean variance approach.

Efficient Portfolios

- The standard myopic optimization of an expected utility function defining the efficient portfolio, the efficiency frontier, the implied performance and the implied stochastic discount factor is reviewed.
- A portfolio is defined by its allocation θ_o, θ_1,. . ., θ_n in the different assets. The portfolio value at date t-1 is $W_{t-1} = \theta_o + \theta' e$, where $\theta = (\theta_1, . . ., \theta_n)'$, $e=(1,. . .,1)'$, whereas its future value is: $W_t = \theta_o(1+R^f_t) + \theta'(e+\check{R}_t)$, where $\check{R}_{1t, ,,,,,,} \check{R}_{nt}$
- The utility function may represent the preferences of the individual investors, and the comparison of a market portfolio with the set of efficient individual portfolios may be a basis for testing equilibrium models, whenever a K-fund separation theorem is satisfied.

Optimization Problem

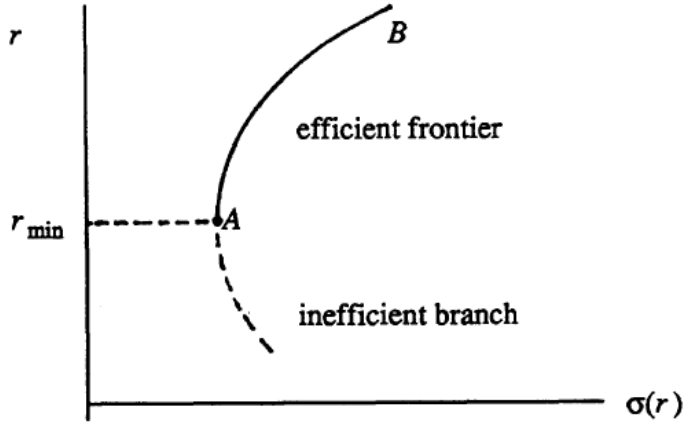
- Let us introduce an (indirect) utility function u(A;w), depending on some (multidimensional) parameter A, and satisfying the standard assumptions of increasingness and concavity with respect to w. The optimization problem is:

- $\max_{\theta o,\, \theta} Eu(A;W_t)$

  s.t.: $W_{t-1} = \theta_o + \theta' e$

- **It is important to note that the special structure of the set of mean-variance efficient portfolios provides the basis for the Capital Asset Pricing Model**

- **In practical applications additional constraints sometimes have to be imposed, such as exclusion of short sales, bounds on the weights of individual assets, etc. With constraints of this type, the optimization problem is still quadratic convex and powerful numerical methods are available. However, in general the special structure of the efficient frontier is destroyed.**



Multifund separation hypothesis

- **Two-fund separation hypothesis**: If there is a risk-free asset, we have to check if the set of efficient portfolios is such that the allocation in risky assets θ(β) is included in a one-dimensional subspace. The null hypothesis is:

$$H_{o,2} = \{\exists \theta^1 \in \mathbb{R}^n, \forall \beta, \exists \mu(\beta) \in \mathbb{R} : \theta(\beta) = \mu(\beta)\theta^1\}.$$

- The hypothesis $H_{0,2}$ can be considered as a spanning hypothesis, in which the set of generating portfolios is not a priori given. This problem is standard in a mean variance framework. It is considered here in the general framework of expected utility maximization.

- **K-fund separation hypothesis**: The null hypothesis becomes:

$$H_{o,K} = \{\exists K \in \mathbb{N}^*, \exists \theta^1, \ldots, \theta^{K-1} \in \mathbb{R}^*, \forall \beta, \exists \mu_1(\beta), \ldots, \mu_{K-1}(\beta) \in \mathbb{R} : \theta(\beta) = \mu_1(\beta)\theta^1 + \ldots + \mu_{K-1}(\beta)\theta^{K-1}\}.$$

- Even if it is still possible to follow a two-step approach by first estimating K, θ^1,..,.θ^{K-1}1, $\mu_1(.)$,. . ., $\mu_{K-1}(.)$ and then computing a test statistic, the asymptotic properties of the corresponding estimators and test statistics are difficult to derive.

Application

- We consider the daily Hang Sen Index (HSI) over the period January 2nd, 1997, to January 29th, 1999, corresponding to 501 observations. The HSI is a value-weighted index of 33 stocks highly traded on the Hong Kong Stock Exchange, which accounts for about 70% of total market capitalization. The Hong Kong Stock Exchange is the second largest Asian market after Tokyo.

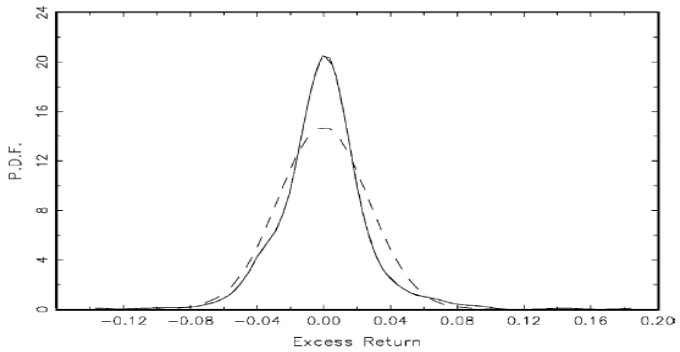


Fig. 5. P.D.F. of the index excess return (daily).

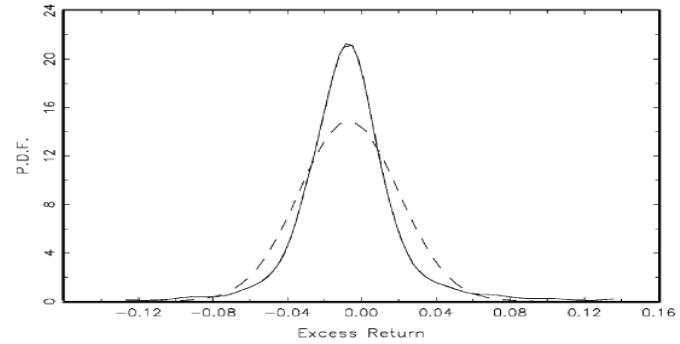


Fig. 6. P.D.F. of the call excess return (daily, $t = 1$ month).

- In Figs. 5 and 6, the dotted curves are the P.D.F.'s of the gaussian distribution with the same mean and variance as the other P.D.F. plotted in the same figure. The kurtosis are 10.9 for the index return and 9.1 for the call return, respectively.

- Indeed, at equilibrium in the usual mean variance framework, the expected excess return is generally positive due to the risk premium effect. However, this result assumes gaussianity or at least a mean variance framework. However, an asset with negative expected return can be demanded by investors if its distribution is skewed especially for the extreme risk. Our example is typically of this kind because an option is introduced to hedge a risk of a given sign; an option is an "insurance" product which can have a cost in terms of expected return.

Application

- For each investment horizon, we consider the optimal allocations for a portfolio including the riskless asset, the basic risky asset (i.e., the index) and the call written on the index. The mean variance optimal allocation and the optimal allocation based on a CARA utility function are both displayed as function of the risk aversion parameter.
- In Figs. 13 and 14, the dotted lines correspond to the mean variance allocations and the solid line to the CARA utility based allocations;For instance, the Basle Committee suggested a historical simulation of the Value at Risk to account for asymmetry and tails, instead of a mean variance approach, which would strongly underestimate the required capital.

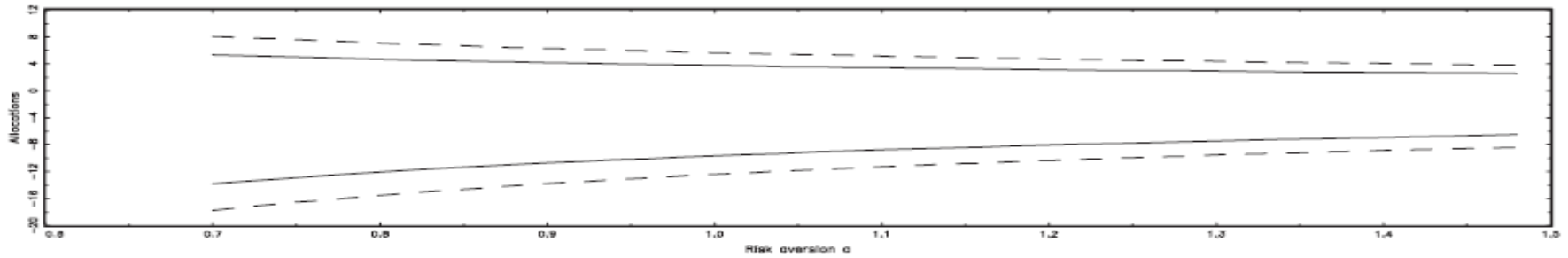


Fig. 13. Optimal portfolios: expected utility and mean variance (daily).

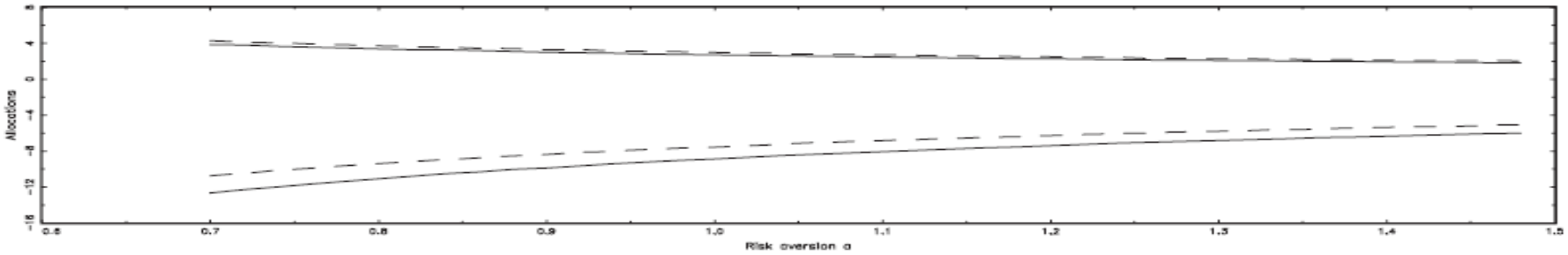


Fig. 14. Optimal portfolios: expected utility and mean variance (3 months).

Conclusion

- If the returns were gaussian, the mean variance and the CARA utility-based allocations would be identical. Figs. 13 and 14 show that this is not the case.
- The aim of this paper was to develop an approach of efficiency analysis in a general framework of expected utility maximization.
- We have seen that the standard approaches proposed in the gaussian mean variance framework can be extended to the general case. They include the estimation of efficient portfolios, of performances, of implied risk aversion and implied stochastic discount factors as well as tests of various efficiency or fund separation hypothesis, or estimation of spanning sets of portfolios.

In this context in our reference paper, the portfolio problem which is identified very simply in the modern portfolio theory, is tried to be extended as multifund seperation hypothesis.

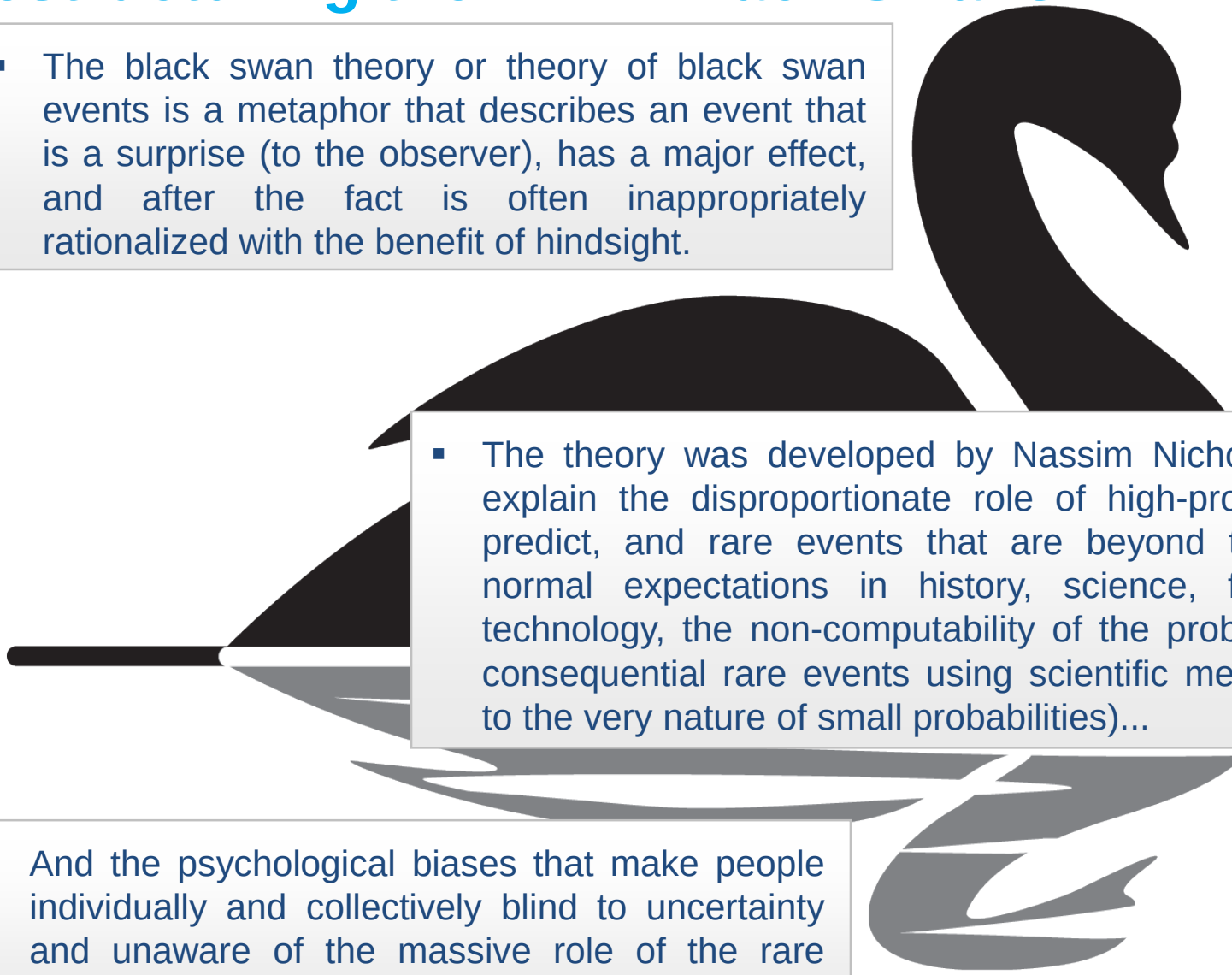
Comments and expanded literature

- The parameters embadded as A in the utility function $Eu(A;W_t)$ was not explanied detailly in the paper. The structure of the A is realted with the attitude of the investor whether he/she is risk averse or risk seeking.
- In the 6th Chapter of Bodie, Kane and Marcus' Investment, this subject is explained very detailly.
- The \market model" version of the CAPM is formally subsumed by the 1-factor APT, and Steve has argued persuasively that most tests of the CAPM implicitly use the one-factor APT as the null hypothesis. Steve (JET 1978) also showed that having a factor structure for returns is a suffcient condition for K-fund separation (i.e., all risk averse agents would be happy to hold portfolios of the K-mutual funds instead of a general portfolio).

- Konno, Shirakawa, and Yamazaki (1993) formulate a general portfolio optimization problem maximizing skewness subject to fixed expected return and variance constraints, whereby both the quadratic and cubic terms are linearly approximated to yield a mean-absolute deviation-skewness model.
- Boyle and Ding (2005) pick up from there and check the effect of only using a linear approximation for the cubic term.

Restructuring the MPT: Black Swans

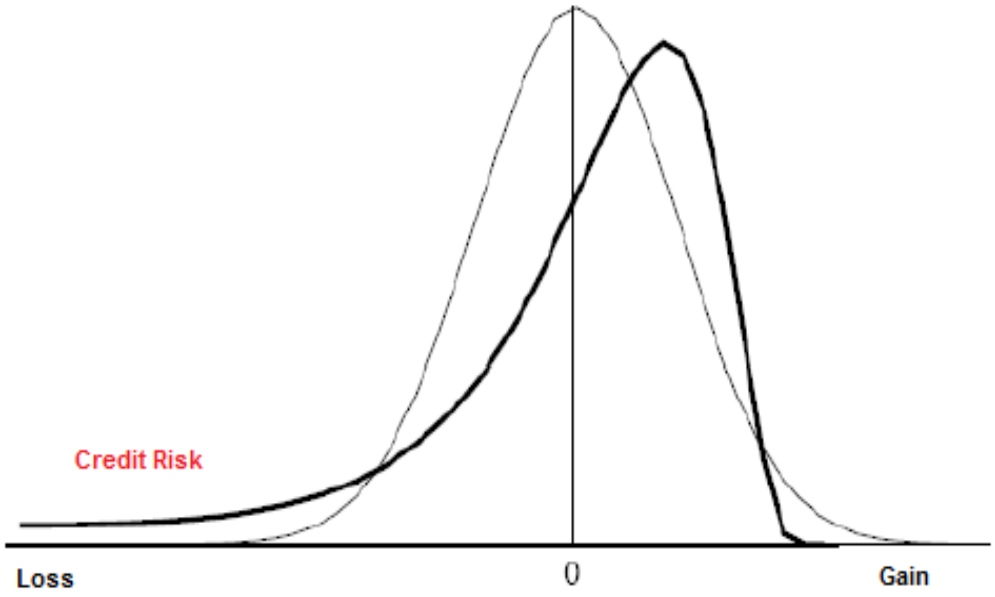
- The black swan theory or theory of black swan events is a metaphor that describes an event that is a surprise (to the observer), has a major effect, and after the fact is often inappropriately rationalized with the benefit of hindsight.



- The theory was developed by Nassim Nicholas Taleb to explain the disproportionate role of high-profile, hard-to-predict, and rare events that are beyond the realm of normal expectations in history, science, finance, and technology, the non-computability of the probability of the consequential rare events using scientific methods (owing to the very nature of small probabilities)...

- And the psychological biases that make people individually and collectively blind to uncertainty and unaware of the massive role of the rare event in historical affairs

Restructuring the MPT: Black Swans



A fat-tailed distribution is a probability distribution that has the property, along with the heavy-tailed distributions, that they exhibit extremely large skewness or kurtosis.

The Black–Scholes model of option pricing is based on a normal distribution. If the distribution is actually a fat-tailed one, then the model will under-price options that are far out of the money, since a 5 or 7 sigma event is much more likely than the normal distribution would predict.

- Black Swans of the last 15 years is plotted in the graph on right handside
- Latest Black Swan was observed on March 2011 in Japan.